

Percolation on dual lattices with k -fold symmetry

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Abstract

Zhang found a simple, elegant argument deducing the non-existence of an infinite open cluster in certain lattice percolation models (for example, $p = 1/2$ bond percolation on the square lattice) from general results on the uniqueness of an infinite open cluster when it exists; this argument requires some symmetry. Here we show that a simple modification of Zhang's argument requires only 2-fold (or 3-fold) symmetry, proving that the critical probabilities for percolation on dual planar lattices with such symmetry sum to 1. Like Zhang's argument, our extension applies in many contexts; in particular, it enables us to answer a question of Grimmett concerning the anisotropic random cluster model on the triangular lattice.

1 Introduction and results

Let G be an infinite, locally finite graph, and $0 < p < 1$ a real parameter. In *independent bond percolation* on G , each edge of G is assigned a *state*, *open* or *closed*; the states of the edges are independent, and each edge is open with probability p . An *open cluster* is a maximal connected subgraph of G all of whose edges are open, i.e., a component of the *open subgraph* formed from G by deleting the closed edges. We write \mathbb{P}_p for the corresponding probability measure.

Writing $\theta(p) = \theta_G(p) = \theta_{G;v}^b(p)$ for the probability that a fixed 'starting vertex' v is in an infinite open cluster, the (Hammersley) *critical probability* $p_H = p_H^b(G)$ for bond percolation on G is defined by

$$p_H^b(G) = \inf\{p : \theta_G(p) > 0\},$$

where v is any fixed vertex of G . (There are other natural notions of critical probability. For well behaved graphs, including the lattices studied here, these

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coincide, and one writes p_c^b for their common value. Here we are concerned only with p_H^b .) Let E_∞ denote the event that there is an infinite open cluster somewhere in G . Kolmogorov's 0/1-law implies that $\mathbb{P}_p(E_\infty)$ is either 0 or 1 for any p . Simple standard arguments show that $\mathbb{P}_p(E_\infty) = 0$ for $p < p_H$, and $\mathbb{P}_p(E_\infty) = 1$ for $p > p_H$.

By a *plane lattice* we shall mean a (multi-)graph G drawn in the plane with non-crossing edges, with the following properties: G is connected, infinite, and locally finite, the vertex set $V(G)$ is a subset of \mathbb{R}^2 containing no accumulation points, and there are independent vectors $w_1, w_2 \in \mathbb{R}^2$ such that the translations T_{w_i} of \mathbb{R}^2 through the vectors w_i induce isomorphisms of G as a plane graph. Usually, G is drawn with straight edges, as in the examples in Figure 1, but we shall not require this. However, we shall assume, as we may, that the edges of G are drawn as piecewise linear curves.

Note that the vertex set of a plane lattice G need not be a lattice in the algebraic sense. However, for every vertex v of G there is an isomorphism of G mapping v to a vertex v' in the parallelogram with corners $0, w_1, w_1 + w_2, w_2$, so $V(G)$ is the union of finitely many translates of the lattice $\{a_1 w_1 + a_2 w_2 : a_1, a_2 \in \mathbb{Z}\}$.

For $k \geq 2$, we say that a plane lattice G has *k-fold symmetry* if the rotation about the origin through an angle of $2\pi/k$ maps the plane graph G into itself. For example, the square lattice has 4-fold symmetry. The hexagonal lattice has 6-fold symmetry as long as the origin is chosen appropriately, i.e., as the centre of a hexagon. Figure 1 shows two plane lattices; the one on the left has 2-fold symmetry, the one on the right does not have k -fold symmetry for any $k \geq 2$.

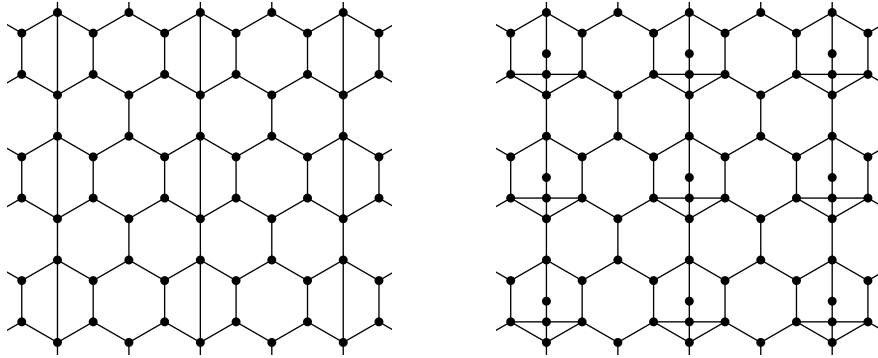


Figure 1: Two plane lattices obtained by ‘decorating’ the hexagonal lattice. The lattice on the left has 2-fold symmetry if the origin is chosen suitably. The lattice on the right has mirror symmetry, but does not have k -fold symmetry for any $k \geq 2$.

The *dual* G^* of a plane (multi-)graph G is the (multi-)graph with one vertex for every face of G , and an edge e^* for every edge of G ; the edge e^* joins the vertices of G^* corresponding to the faces of G in whose boundary e lies. It is easy to see that, if G is a plane lattice, then so is an appropriate drawing of

G^* . If, in addition, G has k -fold symmetry, then G^* may be drawn with k -fold symmetry.

Aizenman, Kesten and Newman [2] showed that, under rather mild conditions, the probability that there is more than one infinite open cluster is zero; a little later, Burton and Keane [5] gave a very simple proof of this result. This result applies in particular to independent bond percolation on a plane lattice (as well as to lattices in all dimensions). In 1988, Zhang found a simple way to deduce that if G is a plane lattice with k -fold symmetry, $k \geq 4$, and $0 < p < 1$, then one cannot have both $\theta_G(p) > 0$ and $\theta_{G^*}(1-p) > 0$; this argument was first published in Grimmett [7, p. 195]. Zhang's argument uses only symmetry, Harris's positive correlation lemma, and the Aizenman–Kesten–Newman result. For the square lattice, which is self-dual, it follows that $\theta(1/2) = 0$, giving an alternative proof of this old result of Harris [11]. Using, for example, the exponential decay result of Menshikov [14], it follows that $p_H(\mathbb{Z}^2) = 1/2$; see [9] or [4, Section 5.2]. The advantage of Zhang's argument over the (by now) many other proofs of this celebrated result of Kesten [12] (see [9, 3, 4], for example) is that it does not rely on a 'Russo–Seymour–Welsh' type lemma [15, 17], and so adapts easily to other lattices, and also to other models, such as the random cluster model of Fortuin and Kasteleyn [6].

Our aim is to show that, contrary to the generally held belief (see Grimmett [9], for example), it is fairly easy to adapt Zhang's argument to weaken the symmetry requirement: k -fold symmetry for any $k \geq 2$ is enough.

Theorem 1. *Let G be a plane lattice with k -fold symmetry for some $k \geq 2$, and let $0 < p < 1$. Then either $\theta_G(p) = 0$ or $\theta_{G^*}(1-p) = 0$.*

We prove this result in the next section; in the final section we shall discuss some extensions and an application.

Before turning to the proof, let us remark that, in a paper published after the first draft of this paper was written, Sheffield [16, Theorem 9.3.1] has proved a result corresponding to Theorem 1, but applying to all plane lattices, with no additional symmetry requirement. His proof of this stronger result is rather involved.

2 The proof

As usual in percolation, we declare an edge e^* of G^* to be open if and only if e is closed. Note that the states of the edges of G^* are independent, and each edge is open with probability $1-p$.

By an *open path* in G we mean a finite or infinite path $P = v_0 e_0 v_1 e_1 v_2 \cdots$ in the graph G such that every edge e_i of P is open. An open path in G^* is defined similarly. Since each edge of G (or G^*) is a piecewise linear curve in the plane, so is any open path P .

Let D_R be the closed disc of radius R centred at the origin. We say that an infinite open path P in G or G^* *leaves D_R at the point x* if, when P is viewed as a curve in the plane, the last point of P in D_R is x , i.e., if P contains an infinite

piecewise linear path meeting D_R only at x . If the boundary of D_R is divided into arcs A_1, \dots, A_r , then we write $L_i = L_i(R; A_i)$ for the event that there is an infinite open path in G leaving D_R at some point $x \in A_i$. Similarly, we write L_i^* for the event that there is an infinite open path in G^* leaving D_R at some point of A_i . Note that each L_i is an *increasing* event: if L_i holds and we change the states of one or more edges from closed to open, then L_i still holds.

The essence of Zhang's argument is as follows. Let G be a plane lattice with k -fold symmetry, $k \geq 4$, and suppose that $\theta_G(p) > 0$ and $\theta_{G^*}(1-p) > 0$. From Kolmogorov's 0/1-law, with probability 1 both G and G^* contain infinite open clusters. As $R \rightarrow \infty$, the discs D_R increase to cover \mathbb{R}^2 , so the probability that D_R meets an infinite open cluster in G tends to 1, i.e., for any $\varepsilon > 0$, if R is large enough this event has probability at least $1 - \varepsilon$. Divide the boundary of D_R symmetrically into k arcs A_1, \dots, A_k . If D_R meets an infinite open cluster in G , then there is an infinite open path in G leaving D_R at some point x , which must lie in one of the A_i . Thus, the union $L_1 \cup \dots \cup L_k$ has probability at least $1 - \varepsilon$. Since the L_i are increasing, it follows from Harris's Lemma (see below) that $\mathbb{P}(L_i) \geq 1 - \varepsilon^{1/k}$ for some i . But $\mathbb{P}(L_i) = \mathbb{P}(L_j)$ by symmetry. Thus, if R is large enough, we have $\mathbb{P}(L_i) \geq 1 - \varepsilon^{1/k}$ for every i . Similarly, if R is large enough, $\mathbb{P}(L_i^*) \geq 1 - \varepsilon^{1/k}$ for every i .

Taking $\varepsilon < (1/4)^k$, we find that with positive probability the event $L_1 \cap L_2^* \cap L_3 \cap L_4^*$ holds. But simple topology shows that when this event holds, one of G and G^* contains two infinite open clusters. This contradicts the Aizenman–Kesten–Newman Theorem.

To adapt this argument to the case $k = 2$ or $k = 3$, we simply divide the boundary of D_R into $2k$ arcs, in a way that has k -fold symmetry. Roughly speaking, by moving the division point between A_1 and A_2 appropriately, we can ensure that L_1 and L_2 (and hence, by symmetry, all L_i) have about the same probability, so $\mathbb{P}(L_1)$ is close to 1. With the arcs now fixed, there is *some* i with $\mathbb{P}(L_i^*)$ close to 1, and hence $\mathbb{P}(L_{i+2}^*) = \mathbb{P}(L_i^*)$ also close to 1, so $L_i^* \cap L_{i+1} \cap L_{i+2}^* \cap L_{i+3}$ has positive probability.

For a reader familiar with Zhang's argument, it is a straightforward exercise to turn the above outline into a complete proof of Theorem 1; nevertheless, we shall spell out the details in full.

Proof of Theorem 1. Let G be a plane lattice with k -fold symmetry, and let G^* be its dual, drawn as a plane lattice with k -fold symmetry. Let $0 < p < 1$ be fixed, and suppose for a contradiction that $\theta_G(p) > 0$ and $\theta_{G^*}(1-p) > 0$. We assume for convenience that G and G^* are drawn with all edges piecewise linear. Rotating the coordinate system, if necessary, we may assume that no line segment making up a part of an edge of G or G^* is parallel to the x -axis. Thus, the set B_1 of $R > 0$ such that the point $(R, 0)$ lies on an edge of G or G^* is countable. As $V(G)$ is countable, the set B_2 of $R > 0$ such that any vertex of G or G^* lies on the boundary of D_R is also countable. From now on we only consider $R \notin B = B_1 \cup B_2$.

Given $R \notin B$ and $0 \leq \varphi \leq 2\pi/k$ such that $Q_1 = (R \cos \varphi, R \sin \varphi)$ does not lie on an edge of G or G^* , let A_1 be the boundary arc of D_R from $Q_0 = (R, 0)$ to Q_1 ,

and let A_2 be the boundary arc of D_R from Q_1 to $Q_2 = (R \cos(2\pi/k), R \sin(2\pi/k))$. For $i \geq 3$, let A_i and Q_i be defined by rotating A_{i-2} and Q_{i-2} anticlockwise about the origin through an angle $2\pi/k$; when referring to A_i or Q_i , we shall always take the subscript modulo $2k$. Since no Q_i lies on an edge of G or G^* , it will be irrelevant whether A_i includes its endpoints; if we include one endpoint into each A_i , then the $2k$ arcs A_i partition the boundary of D_R .

Let $L_i = L_i(\varphi) = L_i(R, \varphi)$ be the event that G contains an infinite open path leaving D_R at a point of A_i , and let $L_i^* = L_i^*(\varphi) = L_i^*(R, \varphi)$ be defined similarly for G^* . Again, we take the subscript modulo $2k$. Note that $\bigcup_i L_i$ is the event that D_R meets an infinite open cluster in G .

Since $\theta_G(p) > 0$, the probability that G contains an infinite open cluster is positive, and hence (by Kolmogorov's 0/1-law, for example) equal to 1. Thus, $\mathbb{P}(\bigcup_i L_i) \rightarrow 1$ as $R \rightarrow \infty$. Set

$$\varepsilon = (1 - p)^{2k} / 5^{2k} > 0, \quad (1)$$

noting that ε depends on p but not on R or φ . Then, if R is large enough, we have

$$\mathbb{P}\left(\bigcup_i L_i\right) \geq 1 - \varepsilon. \quad (2)$$

Similarly, since $\theta_{G^*}(1 - p) > 0$, if R is large enough we have

$$\mathbb{P}\left(\bigcup_i L_i^*\right) \geq 1 - \varepsilon. \quad (3)$$

For the rest of the proof, we fix an $R \notin B$ such that (2) and (3) hold.

The events L_i are increasing, so their complements $\overline{L_i}$ are decreasing. Harris [11] showed that, if E_1 and E_2 are decreasing events in a product probability space, then $\mathbb{P}(E_1 \cap E_2) \geq \mathbb{P}(E_1)\mathbb{P}(E_2)$. As the intersection of two decreasing events is decreasing, it follows that if E_1, \dots, E_j are decreasing, then $\mathbb{P}(\bigcap_i E_i) \geq \prod_i \mathbb{P}(E_i)$. In particular,

$$\mathbb{P}\left(\bigcap_{i=1}^{2k} \overline{L_i}\right) \geq \prod_{i=1}^{2k} \mathbb{P}(\overline{L_i}) = \mathbb{P}(\overline{L_1})^k \mathbb{P}(\overline{L_2})^k,$$

where the equality follows from k -fold symmetry. From (2), it then follows that

$$\mathbb{P}(\overline{L_1})\mathbb{P}(\overline{L_2}) \leq \varepsilon^{1/k}, \quad (4)$$

for any $0 \leq \varphi \leq 2\pi/k$. Arguing in exactly the same way but for G^* , it follows from (3) that

$$\mathbb{P}(\overline{L_1^*})\mathbb{P}(\overline{L_2^*}) \leq \varepsilon^{1/k}. \quad (5)$$

As G is drawn with piecewise linear edges, the set of boundary points of D_R that lie on edges of G is countable. Furthermore, as G is a plane lattice, this set contains no accumulation points, and is thus finite. Let $0 \leq \varphi < \varphi' \leq 2\pi/k$ be such that there is exactly one point x of the arc of D_R from $(R \cos \varphi, R \sin \varphi)$

to $(R \cos \varphi', R \sin \varphi')$ that lies on an edge of G , and suppose that x lies in the interior of this arc. Let L_x be the event that G contains an infinite open path leaving D_R at the point x . Then $L_1(\varphi') = L_1(\varphi) \cup L_x$, and $L_2(\varphi) = L_2(\varphi') \cup L_x$. The events $\overline{L_i(\cdot)}$ and $\overline{L_x}$ are decreasing. Thus, from Harris's Lemma,

$$\mathbb{P}(\overline{L_1(\varphi')}) = \mathbb{P}(\overline{L_1(\varphi)} \cap \overline{L_x}) \geq \mathbb{P}(\overline{L_1(\varphi)})\mathbb{P}(\overline{L_x}).$$

If the unique edge of G on which x lies is closed, then $\overline{L_x}$ holds, so $\mathbb{P}(\overline{L_x}) \geq 1 - p$. Since $L_1(\varphi) \subset L_1(\varphi')$, we thus have

$$\mathbb{P}(\overline{L_1(\varphi)}) \geq \mathbb{P}(\overline{L_1(\varphi')}) \geq (1 - p)\mathbb{P}(\overline{L_1(\varphi)}). \quad (6)$$

Similarly, since $L_2(\varphi) = L_2(\varphi') \cup L_x$, we have

$$\mathbb{P}(\overline{L_2(\varphi')}) \geq \mathbb{P}(\overline{L_2(\varphi)}) \geq (1 - p)\mathbb{P}(\overline{L_2(\varphi')}). \quad (7)$$

As φ is increased from 0 to $2\pi/k$, the probability of $\overline{L_1(\varphi)}$ decreases, while the probability of $\overline{L_2(\varphi)}$ increases. These probabilities change only at a finite set of jumps, corresponding to boundary points of D_R on edges of G , and we have just shown (in (6) and (7)) that, at each jump, $\mathbb{P}(\overline{L_2(\varphi)})$ increases by at most a factor of $(1 - p)^{-1}$, and $\mathbb{P}(\overline{L_1(\varphi)})$ decreases by at most this factor. When $\varphi = 0$ the arc A_1 is empty, so $1 = \mathbb{P}(\overline{L_1(\varphi)}) \geq \mathbb{P}(\overline{L_2(\varphi)})$, while for $\varphi = 2\pi/k$ the arc A_2 is empty and the inequality is reversed. Increasing φ gradually from 0 until $\mathbb{P}(\overline{L_2(\varphi)})$ first exceeds $\mathbb{P}(\overline{L_1(\varphi)})$, it follows that there is some $0 < \varphi < 2\pi/k$ for which

$$\mathbb{P}(\overline{L_1(\varphi)}) \leq \mathbb{P}(\overline{L_2(\varphi)}) \leq (1 - p)^{-2}\mathbb{P}(\overline{L_1(\varphi)}). \quad (8)$$

From now on we fix such a φ , and write L_i for $L_i(\varphi)$, and so on.

Inequalities (8) and (4) imply

$$\mathbb{P}(\overline{L_1}) \leq \sqrt{\mathbb{P}(\overline{L_1})\mathbb{P}(\overline{L_2})} \leq \varepsilon^{1/2k}$$

and

$$\mathbb{P}(\overline{L_2}) \leq \sqrt{(1 - p)^{-2}\mathbb{P}(\overline{L_1})\mathbb{P}(\overline{L_2})} \leq (1 - p)^{-1}\varepsilon^{1/2k}.$$

Since $\mathbb{P}(L_{i+2}) = \mathbb{P}(L_i)$ by symmetry, it follows that for every i we have

$$\mathbb{P}(\overline{L_i}) \leq (1 - p)^{-1}\varepsilon^{1/2k} \leq 1/5,$$

where the final inequality is from (1). In other words, $\mathbb{P}(L_i) \geq 4/5$ for every i .

From (5), there is *some* $j \in \{1, 2\}$ for which $\mathbb{P}(\overline{L_j^*}) \leq \varepsilon^{1/2k} \leq 1/5$. By symmetry, $\mathbb{P}(L_{j+2}^*) = \mathbb{P}(L_j^*) \geq 4/5$. Let $E = L_j^* \cap L_{j+1} \cap L_{j+2}^* \cap L_{j+3}$. Then E is the intersection of four events each of which has probability at least $4/5$, so $\mathbb{P}(E) \geq 1/5 > 0$.

To complete the proof of Theorem 1, we shall invoke the uniqueness result of Aizenman, Kesten and Newman [2] mentioned in the introduction. This result states that, if Λ is a connected d -dimensional lattice, and each edge e of Λ is open

with probability $0 < p_e < 1$ independently of the other edges, then, provided the edge probabilities p_e are preserved by translations of the lattice into itself, the probability that there are two or more infinite open clusters is zero. These assumptions apply both to G and to G^* , so, with probability 1, neither G nor G^* contains two infinite open clusters. Thus, with positive probability, E holds and the infinite open paths P_{j+1}, P_{j+3} in G leaving D_R from A_{j+1} and A_{j+3} are joined by a finite open path P in G , while the infinite open paths P_j^*, P_{j+2}^* in G^* leaving D_R from A_j and A_{j+2} are joined by a finite open path P^* in G^* . Since open paths in G and in G^* cannot cross, this is a topological impossibility. This contradiction completes the proof. \square

3 Consequences and extensions

In this section we give a corollary of Theorem 1, and also some extensions. In all these, the reduced symmetry requirement of Theorem 1 itself is the only new ingredient, so we shall only outline the arguments.

Menshikov [14] proved (essentially) that if G is a lattice, and we take the bonds of G to be open independently with probability $p < p_H^b(G)$, then as the graph distance between two vertices of G increases, the probability that they are joined by an open path decreases exponentially. (This also follows from an independent result of Aizenman and Barsky [1]; see [9] or [4].) A well known consequence of this result is that if G is a plane lattice and G^* is its dual, then $p_H^b(G) + p_H^b(G^*) \leq 1$. (See [4], for example.) Theorem 1 implies that $p_H^b(G) + p_H^b(G^*) \geq 1$, so we obtain the following corollary.

Corollary 2. *Let G be a plane lattice with k -fold symmetry, $k \geq 2$, and let G^* be its dual. Then $p_H^b(G) + p_H^b(G^*) = 1$.* \square

Theorem 1 and Corollary 2 have natural equivalents for site percolation. In *independent site percolation* on a graph G , we take each vertex of G to be open with probability p , independently of the other vertices. Open paths and clusters are then paths and components in the subgraph of G induced by the open vertices, and the definitions of the percolation probability $\theta_G^s(p)$ and critical probability $p_H^s(G)$ are analogous to those of $\theta_G^b(p)$ and $p_H^b(G)$.

There is a well known ‘matching’ relation between certain pairs of graphs that plays a role for site percolation corresponding to plane duality for bond percolation, used by Kesten [13], for example, but going back much earlier. A plane lattice G matches the (not necessarily planar) graph G^\boxtimes obtained from G by adding an edge xy for every pair of non-adjacent vertices x, y of G lying in a common face of G . The relevant properties of G^\boxtimes are that finite open clusters in G or G^\boxtimes are surrounded by closed cycles in G^\boxtimes or G , respectively, and that an open path in G cannot cross a closed path in G^\boxtimes .

Apart from the minor irritation that all references to open paths or clusters in the dual graph (now G^\boxtimes rather than G^*) should be changed to closed paths or clusters, the proofs of Theorem 1 and Corollary 2 carry over almost verbatim to the site percolation setting, to give the following result.

Theorem 3. *Let G be a plane lattice with k -fold symmetry for some $k \geq 2$, and let $0 < p < 1$. Then either $\theta_G^s(p) = 0$ or $\theta_{G^\boxtimes}^s(1-p) = 0$. Furthermore, $p_H^s(G) + p_H^s(G^\boxtimes) = 1$. \square*

Just like Zhang's original argument, the proof of Theorems 1 and 3 carries over to weighted graphs, in which edges are open independently but different edges have different weights, or probabilities of being open. Of course, these probabilities should respect the lattice structure. For a formal statement and proof see [4, Section 5.4]. In this way one can obtain a simple proof of a result of Grimmett [9] (stated, but not proved, by Kesten [13]) giving the 'critical surfaces' for the weighted triangular and hexagonal lattices, in which the weight of an edge depends on its orientation; see [4, p. 152].

Like Zhang's argument, the proof of Theorem 1 (which, after all, is just a simple modification of Zhang's argument) can be applied in contexts other than ordinary percolation. In particular, because the proof relies only on the uniqueness of the infinite cluster, positive correlation of increasing events, and the probability of a single edge being open being bounded away from 1, it also applies to the random cluster model of Fortuin and Kasteleyn [6]. Indeed, Welsh [18] showed that Zhang's argument can be used to prove that the critical probability $p_H(q)$ for the random cluster model on the square lattice satisfies

$$p_H(q) \geq \sqrt{q}/(1 + \sqrt{q}) \quad (9)$$

for all $q \geq 1$ (see also Grimmett [8, 10]); as far as we are aware, this is the only known proof of this result. The proof of Theorem 1 extends this result to the random cluster model on any self-dual plane lattice with k -fold symmetry. More generally, it shows that percolation cannot occur simultaneously in two 'dual' random cluster measures with k -fold symmetry.

The latter observation answers a conjecture of Grimmett [10, relation (6.73)]. For $\mathbf{p} = (p_1, p_2, p_3)$ and $q \geq 1$, let $\varphi_{T, \mathbf{p}, q}^0$ denote the (lower) anisotropic random cluster measure on the triangular lattice, in which edges receive weights p_1, p_2 or p_3 according to their direction, and a weight q is assigned to each cluster; see [10] for a formal definition, which takes a little time to set up. Also, let $\theta_T^0(\mathbf{p}, q)$ denote the probability that the origin is in an infinite open cluster in this measure.

Theorem 4. *Let $q \geq 1$ and $0 < p_1, p_2, p_3 < 1$ be such that*

$$y_1 y_2 y_3 + y_1 y_2 + y_2 y_3 + y_3 y_1 - q \leq 0, \quad (10)$$

where $y_i = p_i/(1-p_i)$. Then $\theta_T^0(\mathbf{p}, q) = 0$.

Proof. As noted by Grimmett [10], the measure $\varphi_{T, \mathbf{p}, q}^0$ is dual to a corresponding measure $\varphi_{H, \mathbf{r}, q}^1$ on the hexagonal lattice, with weights r_i satisfying $r_i/(1-r_i) = q(1-p_i)/p_i$. Grimmett also shows that if equality holds in (10), then, using the star-triangle inequality, this latter measure is equivalent to a measure on the triangular lattice that stochastically dominates $\varphi_{T, \mathbf{p}, q}^0$. We may assume equality in (10) by increasing p_1 , say; the resulting random cluster measure

stochastically dominates the original. It follows that if $\theta_T^0(\mathbf{p}, q) > 0$, then we have percolation in two dual random-cluster measures on lattices with 3-fold symmetry, contradicting the analogue of Theorem 1, whose proof readily adapts to this setting. \square

Let us give one further application of the proof of Theorem 1, proving a result we believe is new. Let $0 \leq r \leq 1$ be fixed, and let T_r be the random triangulation of the plane obtained from the square lattice \mathbb{Z}^2 as follows: add one diagonal to every face, choosing the diagonal parallel to $(1, 1)$ with probability r , and that parallel to $(1, -1)$ with probability $1 - r$, these choices being independent for different faces. If $r = 0$ or $r = 1$, then T_r is isomorphic to the triangular lattice T , which satisfies $p_H^s(T) = 1/2$.

Theorem 5. *Suppose that $0 \leq r \leq 1$. With probability 1, the random triangulation T_r satisfies $p_H^s(T_r) = 1/2$.*

Proof. For $0 \leq r \leq 1$ and $0 < p < 1$, first construct T_r as above, and then select vertices of T_r , i.e., points of \mathbb{Z}^2 , to be open with probability p , independently of each other and of T_r . Let $\mathbb{P}_{r,p}$ denote the associated probability measure, and $\theta(r, p)$ the probability that the origin is in an infinite open path. Our aim is to adapt the proof of Theorem 1, or, rather, of the site percolation version, Theorem 3, to show that

$$\theta(r, 1/2) = 0 \tag{11}$$

for any r .

Any realization of the random graph T_r is a triangulation, and any triangulation G satisfies $G^\boxtimes = G$, so open paths in the random graph T_r cannot cross closed paths in T_r , and so on. Furthermore, when $p = 1/2$ the distribution of the open subgraph of T_r is identical to that of the closed subgraph. Since the model associated to $\mathbb{P}_{r,p}$ has 2-fold symmetry, the proof of Theorem 3 would apply, *mutatis mutandis*, to prove (11), except for one problem: we do not have the required positive correlation. For example, the existence of the single-edge paths $(0, 0)(1, 1)$ and $(0, 1)(1, 0)$ are negatively correlated (indeed, mutually exclusive) events. It turns out that we can get around this.

Let X be the graph obtained from \mathbb{Z}^2 by adding two new vertices $v_{F,1}, v_{F,2}$ for each face F of \mathbb{Z}^2 . Join $v_{F,1}$ to the top-right and bottom-left vertices of F , and $v_{F,2}$ to the other two vertices. Let $v_{F,1}$ be open with probability r , and $v_{F,2}$ with probability $1 - r$, and each vertex of \mathbb{Z}^2 be open with probability p , with all these events independent. When considering events depending only on which vertices of \mathbb{Z}^2 are joined by open paths, the (non-planar) lattice site percolation measure just defined is equivalent to $\mathbb{P}_{r,p}$. Indeed, having examined the states of all vertices of \mathbb{Z}^2 , we need only look within a face F (i.e., consider the vertices $v_{F,i}$ in X or consider the diagonals in T_r) if two opposite vertices of F are open, and the other two vertices closed. (Otherwise, all open vertices of F are joined by paths using edges of \mathbb{Z}^2 .) In this case, the conditional probability that the relevant vertex of X is open is exactly the conditional probability that the relevant diagonal of T_r is present.

Applying Harris' Lemma to site percolation on X , it follows that events such as 'there is an open path in T_r leaving the disc D_R from a certain arc' are positively correlated. As noted above, the proof of Theorem 3 thus goes through, to show that the $\mathbb{P}_{r,1/2}$ -probability that the origin is in an infinite path in T_r is 0. Hence, $p_H^s(T_r) \geq 1/2$ with probability 1. Applying a suitable form of Menshikov's Theorem to the lattice X , it is easy to show that $p_H^s(T_r) \leq 1/2$ with probability 1, and Theorem 5 follows. \square

We close with a question: can one adapt Zhang's argument, or the proof of Theorem 1, to the case of a lattice with mirror symmetry, but without k -fold symmetry for any k ? Past experience suggests that it would be dangerous to conclude that no such variant of the proof exists purely on the basis that we have failed to find one! As noted earlier, Sheffield [16] has proved such a result requiring only the symmetry of a lattice; his proof is far from simple, however.

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